Mathematical Refinement of an Infiltration Equation for Watershed Engineering

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MATHEMATICAL REFINEMENT OF AN INFILTRATION EQUATION FOR WATERSHED ENGINEERING

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c by D. E. Overton[1]

Introduction

In 1961 Holtan (2)[2] proposed ". . .a concept for estimating a potential volume of infiltration from characteristics of the soil and subsequently providing a shape to the curve of progress toward this total." The basic assumption was that infiltration rate be considered as a function of the remaining volume of potential infiltration and the permeability of the confining horizon. Therefore, the time variable would be dependent upon the infiltration rate and the volume increment.

Holtan found that earlier data (8) showed a strong relation between rate of infiltration and the associated potential volume of infiltration to the time of constant rate. He determined a regression equation for each test plot, based upon the average of a number of simulated storm events. The resulting equations were not integrable to arrive at infiltration time functions; therefore, numerical integration was necessary.

In this paper the work of Holtan (2) is reformulated to obtain an integrable equation for analytical solution.

Basic Relationships

For the exploration of the concept Holtan used the data from an infiltrometer survey of two small watersheds in Edwardsville, Ill., 1940-41 (8). The data were taken from plots in which:

(a) Simulated rainfall was applied at the rate of 1.78 inches plus per hour;
(b) Recurrent runs were made;
(c) One type of vegetation predominated;
(d) No unusual condition, such as return flow at some point down slope, was observed.

The experimental equation was:

$$f - f_c = aF_p^{\,n} \qquad [1]$$

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[2]Underscored numbers in parentheses refer to References at end of publication.

where: F = volume of infiltration in inches per hour;
f = (dF/dt) infiltration capacity rate in inches per hour;
f_c = constant infiltration rate in inches per hour, approximated by the permeability of the last permeable horizon;
F_p = potential infiltration to the time of constant rate in inches;
$\underline{a}$ and $\underline{n}$ are constants for a given soil-vegetation complex.

Holtan considered the average value of $\underline{n}$ for all events on each plot and found it to be essentially 1.387 for all plots studied. His $\underline{a}$ values ranged from 0.25 to 0.80. Therefore, his infiltration equation was:

$$f - f_c = aF_p^{1.387} \qquad [2]$$

The available pore space at the beginning of a storm event was not always filled before the infiltration rate became constant. The extent to which this pore space was filled was correlated with vegetative cover.

Thus, the potential infiltration at time (t) = 0 was:

$$(F_p)_{t = 0} = kS_o \qquad [3]$$

where: S_o = available pore space at t = 0, computed as the difference between total pore space and soil moisture in the 0- to 21-inch depth of the soils;
k = percentage factor based on vegetation, which varied from 0.30 for weeds to 1.00 for bluegrass.

For the plots studied, the vegetative factors (k) were determined as follows:

Cover	Factor
Bluegrass	1.00
Crabgrass and alfalfa	.70
Lespedeza and timothy	.45
Alfalfa	.35
Weeds	.30

Formulation for Integration

Integration of equation 2 for an instantaneous infiltration function was not possible. Consequently, the variability from <u>event</u> to <u>event</u> of each plot was reexamined for an integrable expression. The basic concept that infiltration rate is a function of potential infiltration and the permeability of the least permeable horizon was retained.

Equation 1 is integrable only when $\underline{n}$ is an integer. Upon trying the several possibilities for $\underline{n}$, the selected value of 2 was found to best adhere

to the data. Any loss of accuracy induced by substituting 2 for 1.387 in equation 2 was at least partially recovered by a more precise consideration of a values for each event rather than as a plot average. Equation 1 then becomes:

$$f - f_c = aF_p^{\,2} \qquad [4]$$

where a has units of $\text{inch}^{-1}\ \text{hour}^{-1}$.

The basic difference between equations 2 and 4 is that 2 algebraically describes infiltration for each plot, i.e., there will be an average a value for each plot, whereas 4 refers to only one storm event for a given plot and there will be an a value for each event.

The coefficient a must be determined from available data by solving for a in equation 4. The coefficient a can now be found by plotting a range of values of $(f - f_c)$ versus $(F_p{}^2)$ as shown in figure 1 for three storm events on plot 16, Edwardsville, Ill., and determining the slope of the relationship.

On many of the plots there was a strong relationship between a and kS_o (potential infiltration at $t = 0$). Figure 2 shows this relationship for plots 5 and 6 at Edwardsville. Although these relationships are quite good, there were usually but three or four runs per plot available and some scatter appeared on most of them, disallowing definite relationship at this time.

Solution of Infiltration-Time Functions

The basic equations are:

$$f - f_c = aF_p^{\,2} \qquad [4]$$

and

$$kS_o = F_p + F \qquad \text{(continuity)} \qquad [5]$$

Eliminating F_p from equation 4 by substitution from equation 5 and integrating the resulting expression for the boundary condition when $t = 0$, $F = 0$ gives

$$\frac{dF}{dt} = a(kS_o - F)^2 + f_c \qquad [6]$$

The resulting volume-time distribution is

$$F = kS_o - \sqrt{\frac{f_c}{a}}\tan\left[\tan^{-1}\left(\sqrt{\frac{a}{f_c}}\,kS_o\right) - \sqrt{af_c}\;t\right] \qquad [7]$$

The time (t_c) to the constant rate of infiltration (f_c) is

$$t_c = \frac{1}{\sqrt{af_c}} \tan^{-1} \left(\sqrt{\frac{a}{f_c}}\, kS_o \right) \qquad [8]$$

Substituting into equation 8 and simplifying gives

$$F = kS_o - \sqrt{\frac{f_c}{a}} \tan \left[\sqrt{af_c}\, (t_c - t) \right] \qquad [9]$$

The rate equation can then be determined by differentiating equation 9:

$$f = f_c \sec^2 [\sqrt{af_c}\, (t_c - t)] \qquad [10]$$

Testing Fit to Infiltrometer Data

Equations 9 and 10 were applied to the 41 infiltrometer runs (1940-41) from Edwardsville, Ill. (8) and to the 21 infiltrometer runs (1943) from Peoria County, Ill.[3] The Peoria County plots consisted of:

Cover	Soil	Slope of plot (percent)
Bluegrass	Berwick	1
Do.	Muscatine	1 - 2
Do.	Clinton	8
Do.	Viola	8 - 10
Corn	Muscatine	1 - 2
Do.	Clinton	8
Do.	Viola	8 - 10

Figures 3 through 6 summarize the results of testing the fit of the equations to the observed curves. The curves shown were selected from the 62 test cases to illustrate the best fits and also the fits with the greatest deviation from the observed.

Comparisons With Analytical Solutions From Applied Physics

The foregoing equations were empirically derived. From the literature it appears that the most acceptable equations are analytical solutions from applied physical theory. Therefore, it is of interest to compare algebraically the empirically derived equations with the equations of Philip (6,7), Horton (3), Green and Ampt (1), and Kostiakov (4). These analytical equations

[3] Unpublished data.

can also be derived by assuming infiltration rate proportional to either potential infiltration volume (F_p) or to infiltration volume (F).

For example, if we assume that infiltration rate is proportional to infiltration volume and that $n = 1$, then equation 1 would become:

$$f - f_c = aF^{-1} \qquad [11]$$

Upon integrating 11 and applying the boundary condition $t = 0$, and $F = 0$:

$$t = \frac{1}{f_c}\left[F - \frac{a}{f_c}\log\left(1 + \frac{F}{a/f_c}\right)\right] \qquad [12]$$

Equation 12 is algebraically the same as the equation proposed by Philip (6) and Green and Ampt (1). Table 1 summarizes the results of these and other comparisons with analytical infiltration equations from the literature.

TABLE 1.--Comparison of analytical infiltration solutions from applied physical theory with equations derived by assumed proportionality of F_p or F to f

Assumed proportional to f	End rate of f	n	Resulting equation	Analagous to---
F	f_c	-1	$t = \frac{1}{f_c}\left[F - \frac{a}{f_c}\log\left(1+\frac{F}{a/f_c}\right)\right]$	Philip (6): $t = \frac{\mu}{k_H}\left[F-(P+H)(m-m_o)\log\left(1+\frac{F}{(P+H)(m-m_o)}\right)\right]$ Green and Ampt (1): $t = Y\left[F - Z\log\left(1+\frac{F}{Z}\right)\right]$
$F - f_c t$	f_c	-1	$F = \sqrt{2a}\, t^{\frac{1}{2}} + f_c t$	Philip (7): $F - St^{\frac{1}{2}} + At$
F	0	-1	$F = \sqrt{2a}\, t^{\frac{1}{2}}$	Kostiakov (4): $F = bt^{\frac{1}{2}}$
F_p	f_c	1	$F - f_c t + \frac{f_o - f_c}{a}\left(1 - e^{-at}\right)$	Horton (3): $F = f_c t + \frac{f_o + f_c}{K}\left(1 - e^{-Kt}\right)$

Conclusions

The refined infiltration equations 9 and 10 are analytically compact and describe infiltration time distributions for particular storm events on a given soil-vegetation combination. The refined equations fit field infiltrometer data very well.

Holtan's equation and the new analytical expressions derived herein are based upon the premise that infiltration rate is a function of the potential volume of infiltration. The algebraic forms of four well-known infiltration equations originally derived from applied physical theory can also be derived by reformulating the basic relationship defined empirically by Holtan.

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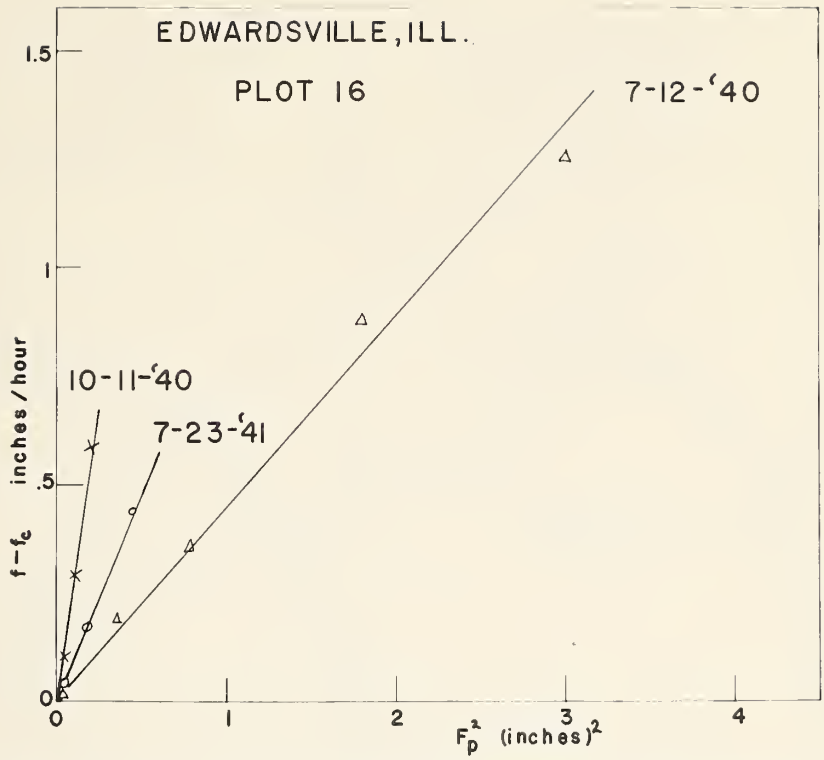

Figure 1.--Determination of the coefficient a for three storm events on plot 16, Edwardsville, Ill.

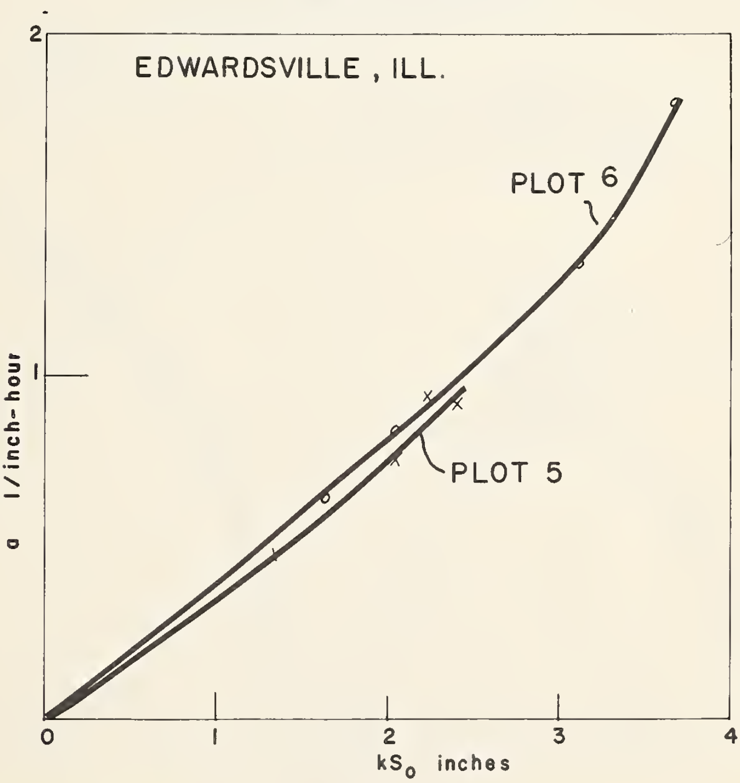

Figure 2.--Total potential infiltration as an indicator of the coefficient a.

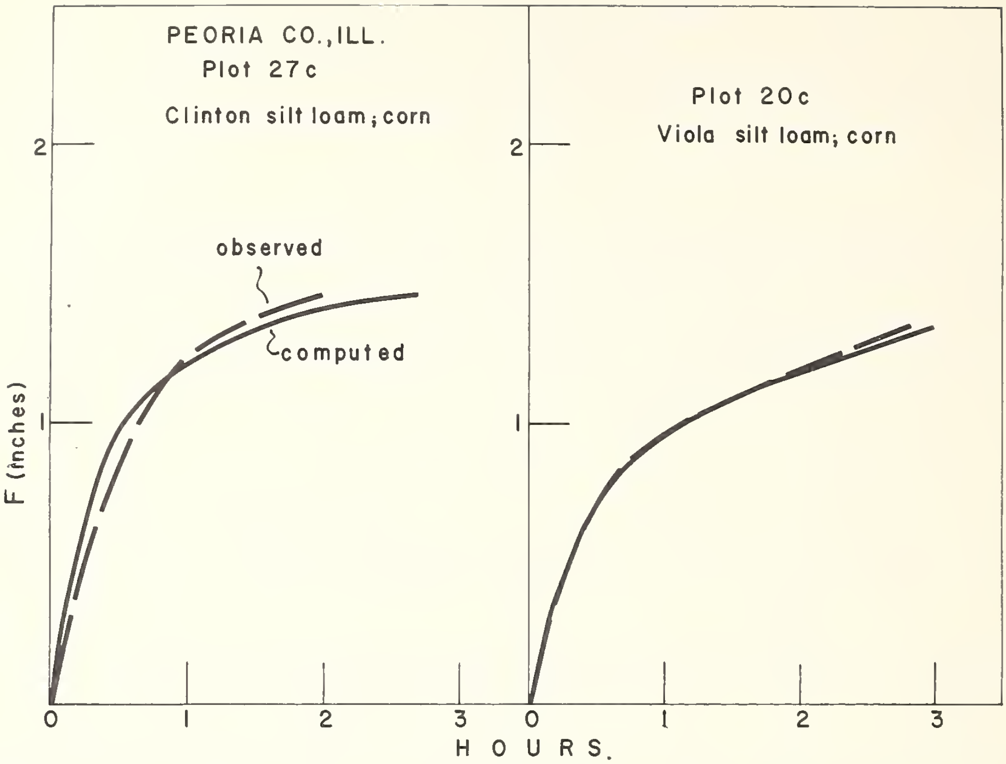

Figure 3.--Observed versus computed infiltration for events on plots 27c and 20c, Peoria County, Ill.

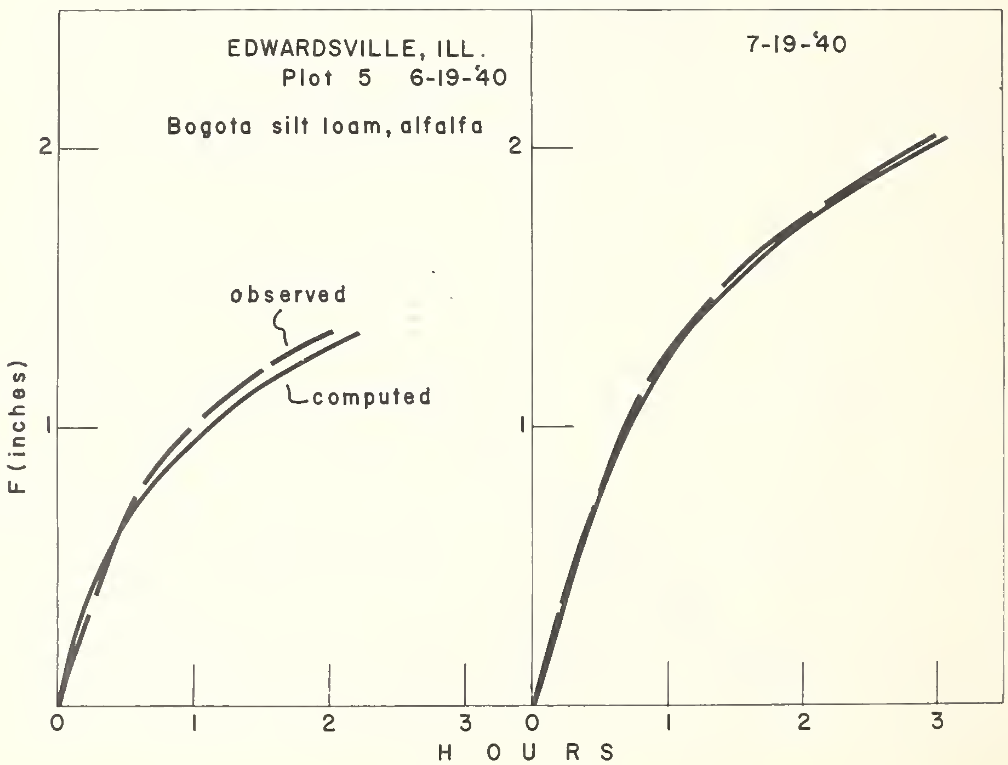

Figure 4.--Observed versus computed infiltration for events on plot 5, Edwardsville, Ill.

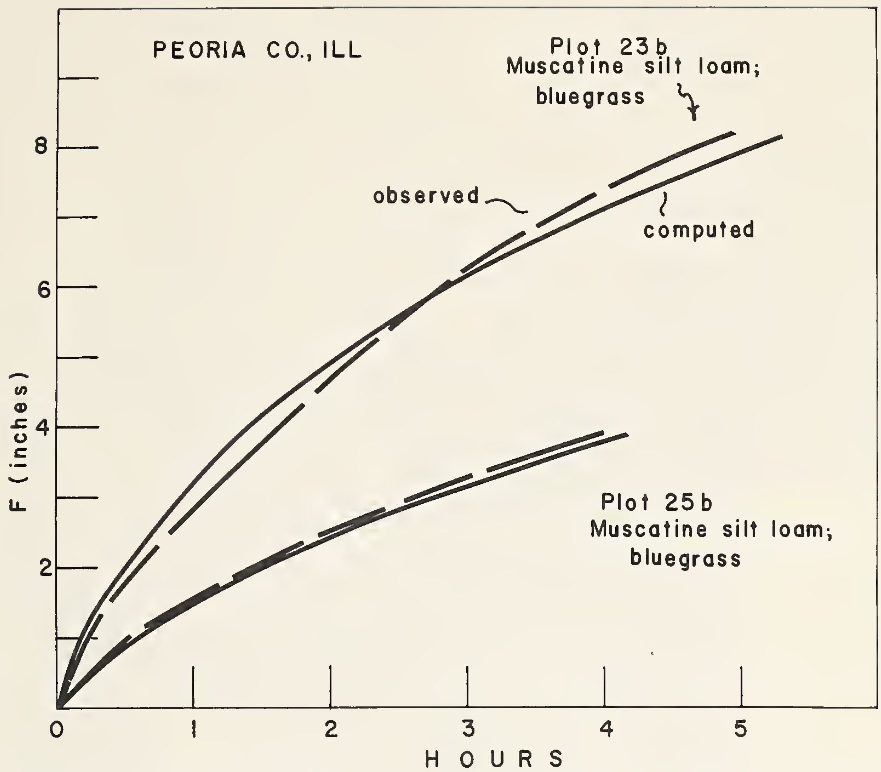

Figure 5.--Observed versus computed infiltration for events on plots 23b and 25b, Peoria County, Ill.

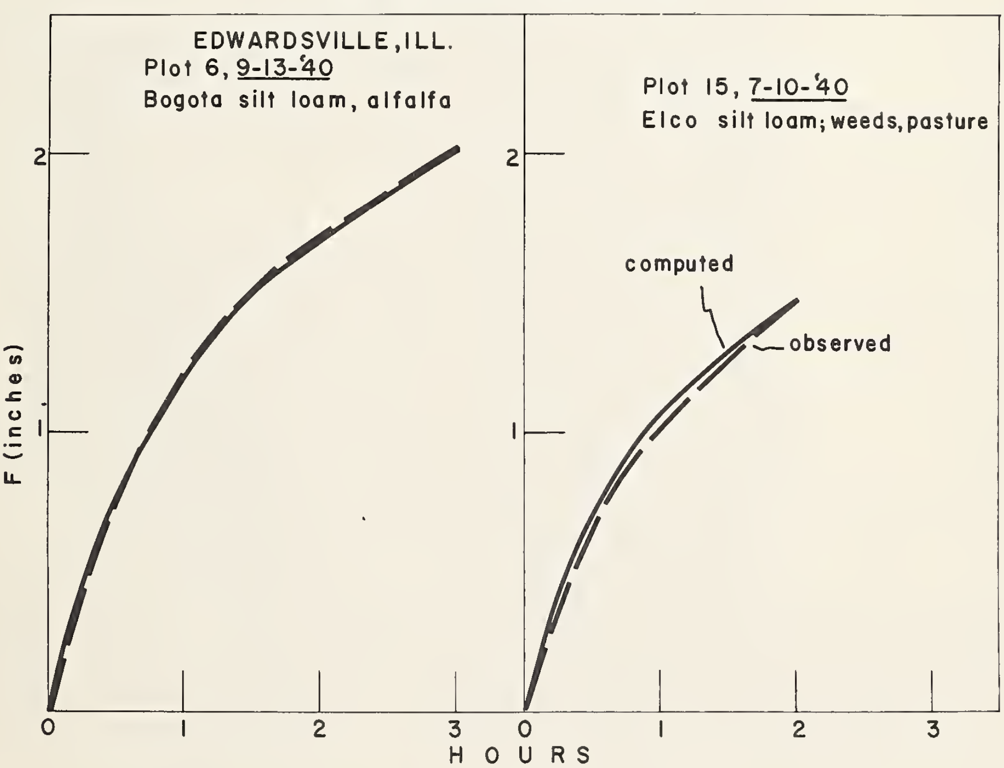

Figure 6.--Observed versus computed infiltration for events on plots 6 and 15, Edwardsville, Ill.

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